Problem 1: Mental Math
NO CALCULATORS.

Example:

Question 1.1:

Question 1.2:

Question 1.3:

Question 1.4:

Question 1.5:

Question 1.6:

Question 1.7:

Question 1.8:

Question 1.9:

Question 1.10:

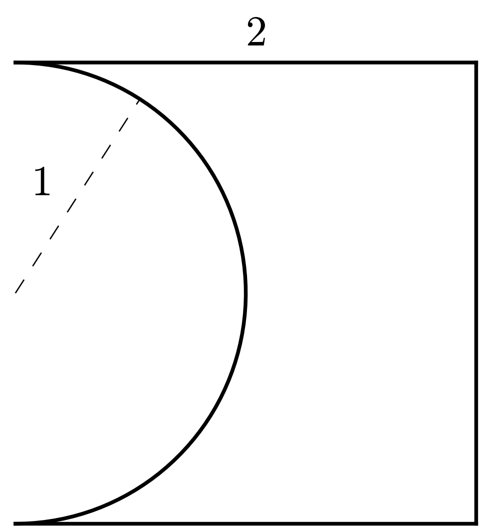
Problem 2: Ponies in the Pasture

10 minutes, no calculators

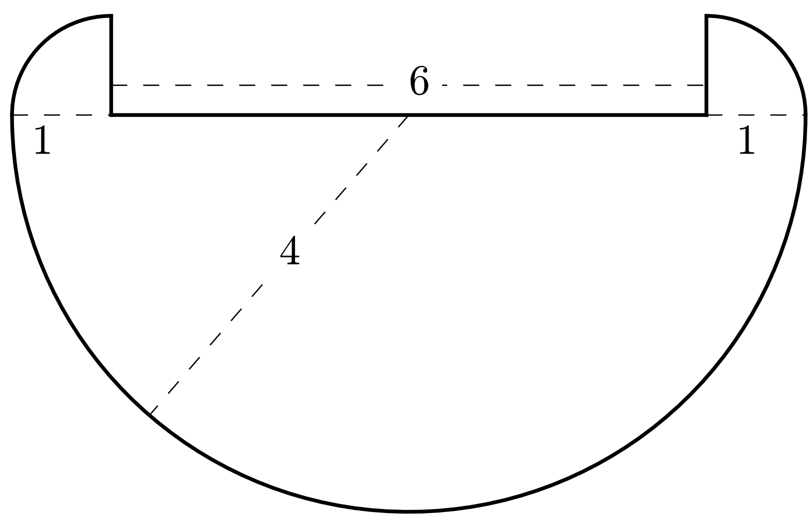
The Exans like to keep ponies as pets. Each pet owner needs to know the size of the available pasture before deciding what kind and what size of pony to get. You should help by finding the areas of the pastures illustrated below. All drawings are to scale, so angles that look like right angles are right angles. All pastures have edges which are either straight lines or portions of circles.

It is OK to leave the letter π in your answer; you don't need to multiply anything by 3.1415926....

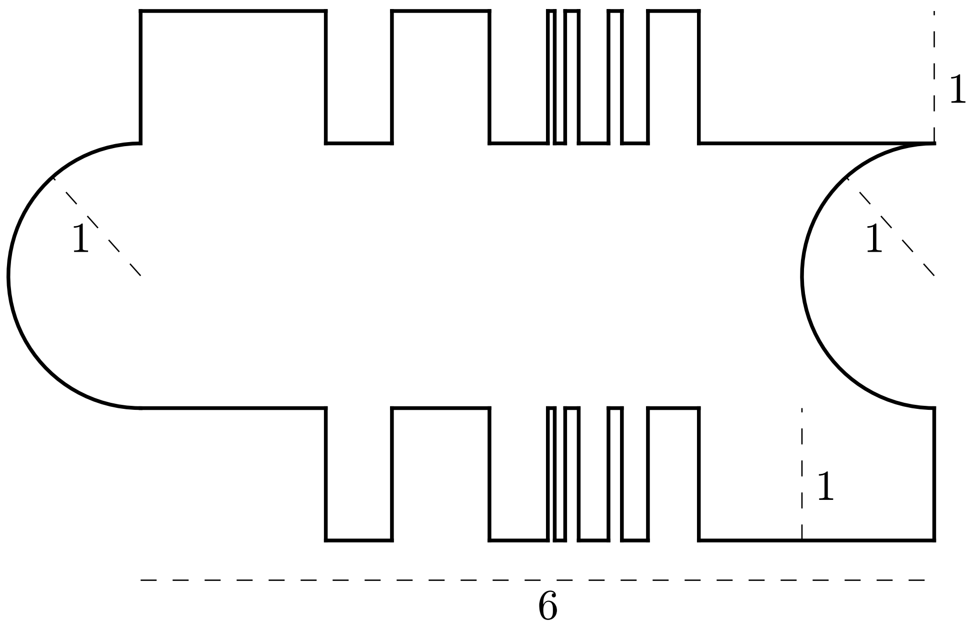
Question 2.1: (3 points)



Question 2.2: (3 points)



Question 2.3: (4 points)



Problem 3: Continued fractions:

10 minutes, no calculators

The Exans have a puzzle that they like to play with. We know how to divide and add, but unlike us the aliens can do this *infinitely* often.

For example an Exan computer can compute

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}}$$

This is called an *infinite continued fraction*. Maybe we can come up with a good guess?

Question 3.1: (4 points) Let $x_1 = 1 + \frac{1}{2}$

And $x_2 = 1 + \cfrac{1}{2+\cfrac{1}{2}}$

And $x_3 = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}$ and so on..

These are *finite continued fractions*.

We can simplify $x_1 = 1 + \frac{1}{2} = 1.5$. Simplify x_2 and x_3 upto two decimal places.

$\boxed{\qquad\qquad\qquad = x_2}$

$\boxed{\qquad\qquad\qquad = x_3}$

We can use a calculator to check that, for instance, $x_6 = 1.414201183\ldots$ and $x_{12} = 1.414213562\ldots$ and if we keep going for long enough our calculator can't tell the difference between the finite continued fractions and the number $\sqrt{2}$!

In fact the Exan computer computes that

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}} \quad = \quad \sqrt{2} \quad !$$

Question 3.2: (3 points) Let's see how we might begin to show something like this without using an Exan computer:

If we are told that there is a number l such that

$$1 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \ddots}}}} \quad = \quad l$$

then we may notice:

$$1 + \cfrac{2}{1 + \overbrace{\boxed{1 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \ddots}}}}}^{=l}} \quad = \quad l$$

And in the gray box is l again! In other words:

$$1 + \frac{2}{1 + l} = l$$

What is the value of the continued fraction, ie. can you solve for l?

$= l$

Question 3.3: (3 points) If we are told

$$1 + \cfrac{4}{2 + \cfrac{2}{3 + \cfrac{4}{2 + \cfrac{2}{3 + \cfrac{4}{2 + \ddots}}}}} \quad = \quad m$$

What is the value of the continued fraction?

Problem 4: Alien Messages

10 minutes, no calculators

When the Exans want to communicate with someone far away, they use red and blue colored flags to transmit their message. So, if Richard wants to send a message to Marcel, he will hold up his red and blue flags in a particular sequence, and Marcel will write down the sequence of colors and then translate them into a sequence of letters.

Part 1:

The alien alphabet consists of the first 16 letters of our alphabet, and each one is encoded as a sequence of flag colors as follows:

$A \to RRRR$	$E \to RBRR$	$I \to BRRR$	$M \to BBRR$
$B \to RRRB$	$F \to RBRB$	$J \to BRRB$	$N \to BBRB$
$C \to RRBR$	$G \to RBBR$	$K \to BRBR$	$O \to BBBR$
$D \to RRBB$	$H \to RBBB$	$L \to BRBB$	$P \to BBBB$

For example, if Richard wants to send Marcel the letter I, he would hold up the flag sequence $BRRR$ (Blue, Red, Red, Red). For longer words, these sequences are chained together, so the sequence $BBRR - RBRR$ translates to the word 'ME'.

Question 4.1: (1 points) What flag sequence would Richard use to send the word 'GO'?

Question 4.2: (1 points) If Marcel receives the sequence $RBBB - RRRR - BBRR$, what word should he translate it to?

Part 2:

Unfortunately, Richard and Marcel occasionally make mistakes by either holding up the wrong flag or writing down the wrong color. To fix this, they've decided to start using seven flags for each letter instead of four. The first four flags are the same as the ones above, but the last three are determined as follows:

We define an operation $\oplus$ so that

$$R \oplus R = R \qquad R \oplus B = B$$

$$B \oplus R = B \qquad B \oplus B = R$$

Now, if the flag colors from the above table are $abcd$, then the new codeword will be $abcdxyz$, where

$$x = a \oplus b \oplus c,$$
$$y = b \oplus c \oplus d, \text{ and}$$
$$z = a \oplus b \oplus d.$$

For example, the old codeword for 'C' was $RRBR$. To find the new codeword, we start with $RRBR\_\_\,\_\_\,\_\_$. Then,

$$R \oplus R \oplus B = B,$$
$$R \oplus B \oplus R = B, \text{ and}$$
$$R \oplus R \oplus R = R,$$

so the last three flags would be BBR. Thus the 7-letter codeword for C is $RRBRBBR$.

Question 4.3: (2 points) What is the 7-letter codeword for J?

If we define the distance between two codewords to be the number of letters that differ between them (for example, the distance between $RRRRRRR$ and $BBBBBBB$ is 7), then the minimum distance between two of the 7-letter codewords described above is three. This allows them to correct one letter per codeword.

Question 4.4: (1 points) Decode the 7-letter code sequence $RBBBRBR - BRRRBRB$.

Part 3:

With this new 7-letter code, Richard and Marcel can always correct a codeword if at most one error occurs. They can find the error as follows:

If they recieve the correct codeword $abcdxyz$, they know that

$$a \oplus b \oplus c \oplus x = R,$$
$$b \oplus c \oplus d \oplus y = R, \text{ and}$$
$$a \oplus b \oplus d \oplus z = R.$$

For any incorrect codeword, some of these sums will equal B instead of R, and that will allow them to determine which symbol was wrong. For example, if the recieved codeword was $BRRRRRR$, we can compute

$$a \oplus b \oplus c \oplus x = B \oplus R \oplus R \oplus R = B,$$
$$b \oplus c \oplus d \oplus y = R \oplus R \oplus R \oplus R = R, \text{ and}$$
$$a \oplus b \oplus d \oplus z = B \oplus R \oplus R \oplus R = B.$$

This tells us that the incorrect symbol appears in the first and third equations, but not the second one. So the incorrect symbol must be in position a, and the corrected codeword is $RRRRRRR$.

Question 4.5: (2 points) The codeword $BRBRBRB$ has exactly one error. What is the correct codeword?

Question 4.6: (3 points) The following code sequence has exactly one error in each 7-letter codeword. Decode the message.

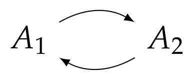
$$BRBRBBR - BBRRBRR - RRBBRRB - RBRRBBR$$

Problem 5: Wandering Aliens

30 minutes, no calculators

The Exans of planet Ex can never decide where they want to live. This means they move around a lot, taking advantage of the great train system that connects the cities of planet Ex. Every day, half the Exans in any city get up and move to whatever city they can get to on the train line.

There are cities, A_1 and A_2, connected by train lines that only run one way. We can represent these trains with arrows, showing the direction that they run:

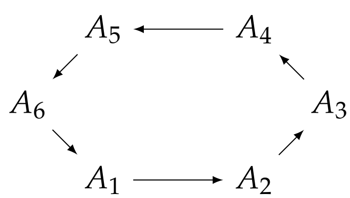


If city A_1 has 100 Exans on a day, then 50 of them will get up and move to city A_2. If city A_2 had 20 Exans on that same day, then 10 of those Exans will move to city A_1. As a result, the next day will start with 60 Exans in city A_1 (50 left and 10 arrived) and 60 in city A_2 (50 arrived and 10 left). Then, 30 from each city will get up and move, and the next there will again be 60 in each.

The city planners of planet Ex noticed that, even though people kept moving, eventually the number of people in each city would be the same every day, because the number of people leaving was the same as the number of people arriving. In order to decide how many houses to build in each city, they want to figure out how many people will be in each city when that number stops changing from day to day, and then build one house per person.

Part 1:

Question 5.1: (4 points) To learn how to do this, they started with a small group of six cities in the state of Ay connected together with train lines that aren't connected to any other cities.



When their test started on day 1, there were 100 Exans in city A_1, 700 in A_2, 900 in A_3, 200 in A_4, 600 in A_5, and 1100 in A_6. As always, half move along the train line. On day 2, how many Exans were in city A_5?

| Exans |

The city planners want to build one house for each Exan once the number of Exans in each city stops changing. How many houses do they need to build in city A_2?

| Houses |

Question 5.2: (6 points) As their understanding of city planning got better, the people on planet Ex decided to add more train lines to their trial system. They noticed that if more than one line was leaving a city, then the Exans split evenly between the lines. That is, if two train lines left a city, then every day $\frac{1}{4}$ (which is $\frac{1}{2}$ of $\frac{1}{2}$) of the population took each train.

$$\begin{tikzcd}
& A_5 \arrow[dl] & A_4 \arrow[l] & \\
A_6 \arrow[dr] & & & A_3 \arrow[ul] \arrow[lll] \\
& A_1 \arrow[r] & A_2 \arrow[ur] &
\end{tikzcd}$$

Again, when they started with there were 100 Exans in city A_1, 700 in A_2, 900 in A_3, 200 in A_4, 600 in A_5, and 1100 in A_6.

How many Exans were in city A_6 on day 2? ______ Exans

How many Exans were in city A_3 on day 2? ______ Exans

The city planners do not want to write down how many Exans are in each city every day! They notice that when they are ready to build houses, the same number of Exans entering a city must be the same number leaving. This is important information! It tells them, for example, that the number of houses in A_2 must be the same as the number in A_1, because half of the Exans from A_1 travel to A_2 just as half of the Exans of A_2 are leaving. On the other hand, one quarter of the Exans of A_3 travel to A_4 while half of the Exans of A_4 leave, so A_3 and A_4 must have a different number of houses.

How many houses will they have to build in city A_1? ______ Houses

How many in city A_4? ______ Houses

Part 2:

In the state of Oh, the Exans are a little more particular about where they live. Instead of half of the Exans in each city leaving each day, in the nicer cities a smaller proportion of the people leave every day.

Question 5.3: (4 points) The first trains were built between two cities, O_1 and O_2. City O_1 was very warm and pleasant, so only $\frac{1}{4}$ of the Exans left every day. City O_2 was a pretty normal place, so $\frac{1}{2}$ of the Exans left, just like in the rest of planet Ex. The city planners of Oh decided to represent this situation with a diagram:

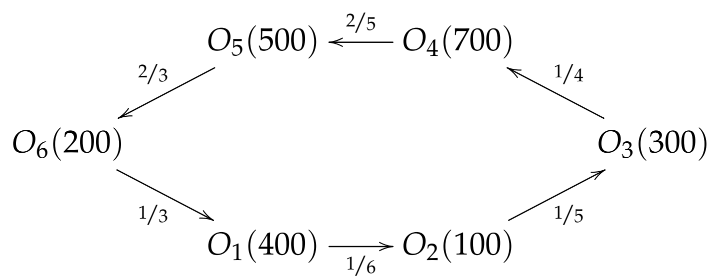
$$\begin{tikzcd}
O_1 \arrow[r, bend left, "\frac{1}{4}"] & O_2 \arrow[l, bend left, "\frac{1}{2}"]
\end{tikzcd}$$

If there was 1000 people in O_1 and 200 people in O_2 when the trains were built, how many Exans were in city O_1 on day 2? ______ Exans

How many houses should they build in city O_1? | Houses

How many houses should they build in city O_2? | Houses

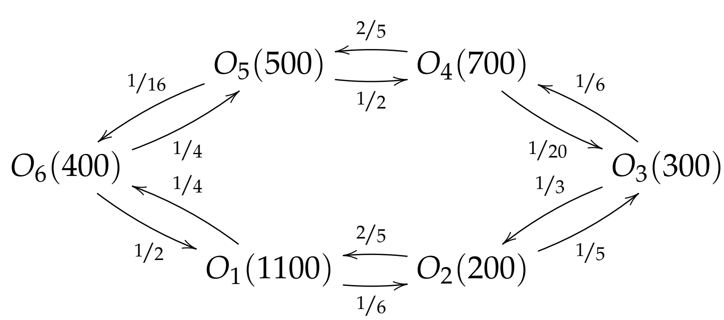
Question 5.4: (4 points) In another part of Oh, six cities were connected with train lines. Again, the numbers next to the arrows tell you what fraction of Exans leave the city and travel along the train line. The numbers in parentheses are the number of Exans who were in each city when the train line was built.



How many houses should they build in city O_3? | Houses

How many houses should they build in city O_6? | Houses

Question 5.5: (6 points) The train company of the state of Oh decided to add some new train lines, and at the same time a bunch of the Exans from the state of Ay moved randomly into Oh to mess up the balance. The city planners decided to demolish all the houses and start over. Not all the train lines have the same level of luxury, so if there are two train lines leaving from a city, one might be more crowded than the other. The numbers next to each arrow tell you what fraction *of the total population of the city* travels along that train line each day. The numbers in parentheses are the number of Exans who were in each city when the trains were built.



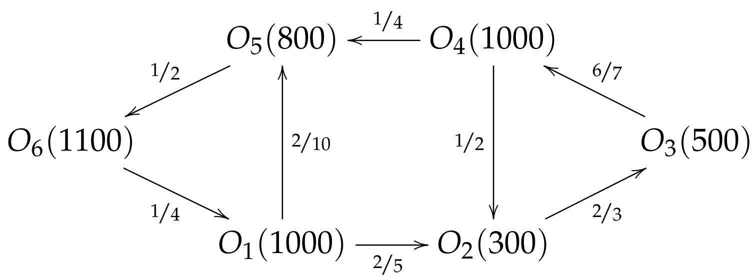
How many houses should they build in city O_6 (disregard any houses built in the previous problem)?

Houses

How many houses should they build in city O_2 (disregard any houses built in the previous problem)?

Houses

Question 5.6: (6 points) Again, the train company changed it's train lines, demolishing some and building some new ones. The city planners decided to demolish the houses and start over again. The numbers next to each arrow tell you what fraction *of the total population of the city* travels along that train line each day. The numbers in parentheses are the number of Exans who were in each city when the trains were built. (Hint: they will need to build 700 houses in city O_3).



How many houses should they build in city O_2 (disregard any houses built in the previous problem)?

Houses

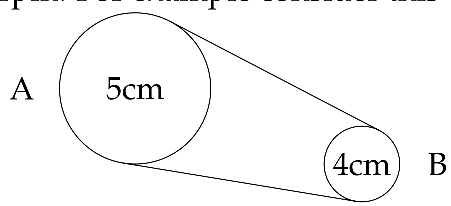
How many houses should they build in city O_6 (disregard any houses built in the previous problem)?

Houses

Problem 6: Ex-Belts

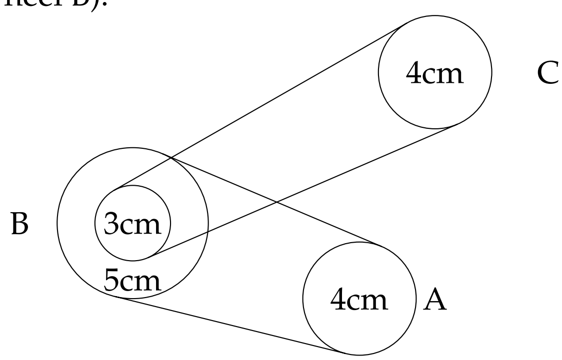
30 minutes, no calculators

On planet Ex all energy is transported by special wheels and belts. Some special wheels are actually two connected wheels, one on top of the other. These two wheels are glued together so they always spin at the same rate. By royal decree, all power sources must turn at exactly 20rpm. For example consider this very simple system:



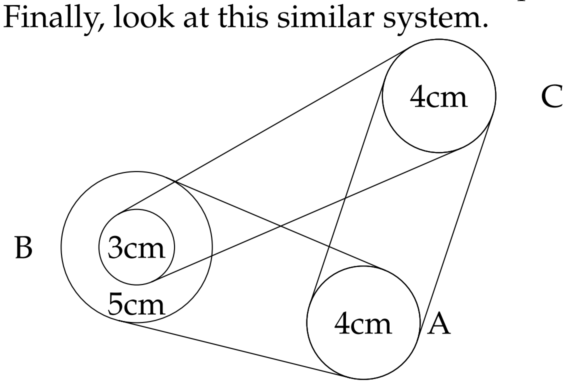
Note that each wheel is marked with it's radius, and the ratio of radii determines how fast wheels spin. So if wheel A is the input wheel, since the radius of A is $\frac{5}{4}$ of the radius of A, B will turn at a rate of $\frac{5}{4} \times 20 = 25$ rpm.

Next look at this system, that uses one of the "two wheels glued together" special wheels (wheel B):



Wheel A is the input wheel, so it must move at 20 rpm. Then wheel B will move at $20 \times \frac{4}{5} =$ 16rpm and wheel C will move at $16 \times \frac{3}{4} =$ 12rpm.

Finally, look at this similar system.

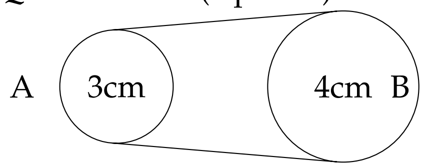


Connecting wheel C directly to wheel A means that wheel A should spin at the same speed as wheel C (because they have the same radius). However, we already saw that wheel C is spinning at 12rpm, and by royal decree wheel A spins at 20rpm. When a wheel has to spin at two different speeds, the system breaks down and all the wheels cannot move.

Part 1:

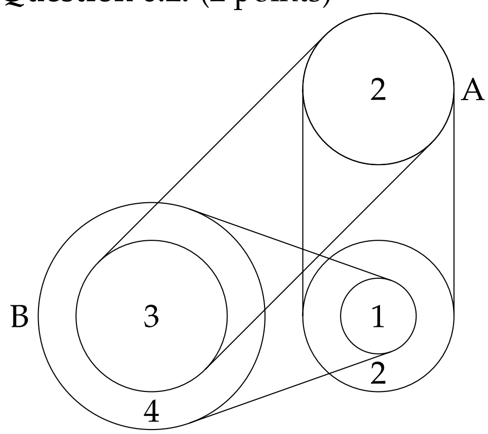
You are the chief plan editor of a new factory that is being built. The factory designer is nice enough to always mark the input wheel with A and the output wheel with B, but he needs your expertise to tell him how fast the wheels are turning. Again the input wheel, wheel A, always moves at 20rpm. If the wheels can't move, enter 0rpm. Write how fast wheel B is turning in the following wheel systems:

Question 6.1: (2 points)



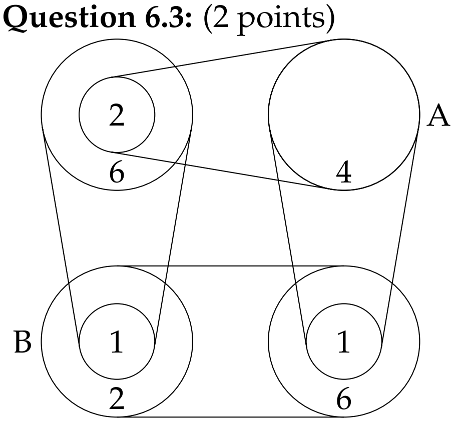
rpm

Question 6.2: (2 points)



rpm

Question 6.3: (2 points)

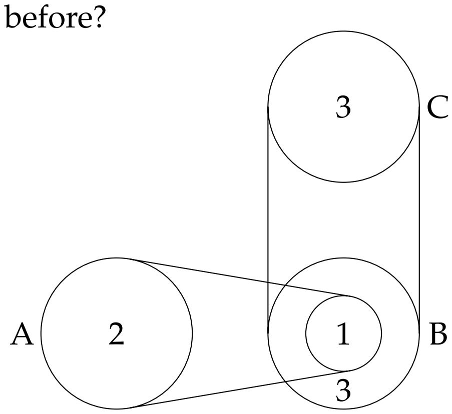


rpm

Part 2:

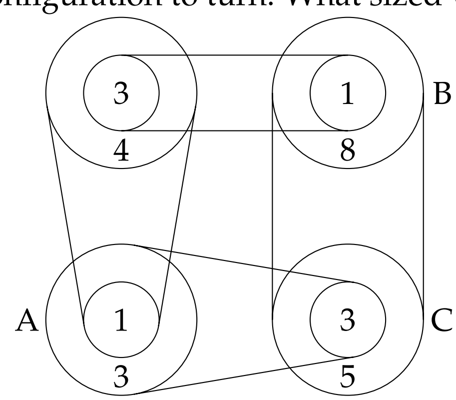
The factory is done being built, and you've been promoted to chief problem fixer!

Question 6.4: (4 points) Wheel B in the following system broke. Unfortunately you don't have any wheels of that size in stock, but you need to get the system running now since it runs the air conditioning in the plant. Without it, you won't be able to take your afternoon nap in comfort! What's another wheel you can replace B with so that wheel C runs at the same speed as before?



inner radius
outer radius

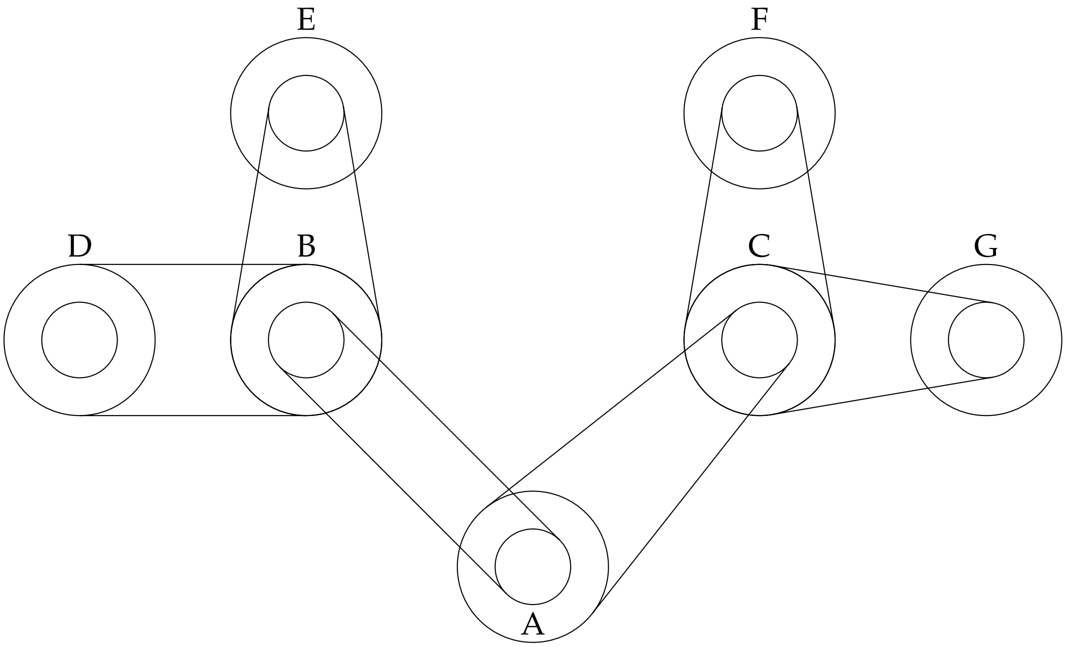
Question 6.5: (5 points) One day while walking through the factory you find that one of the machines isn't working. Eventually, you figure out that someone has switched a wheel out over the weekend and replaced it with one of the wrong size! Now it's impossible for the configuration to turn. What sized wheel can you replace wheel C with to fix the problem?



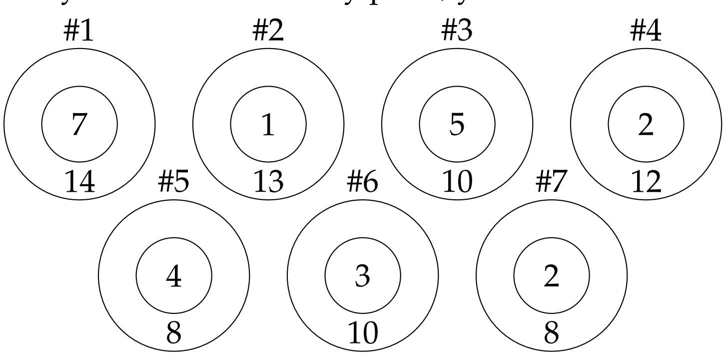
inner radius
outer radius

Part 3:

Question 6.6: (12 points) Working at the factory has some perks. There are plenty of spare wheels lying around for you to use. One day, you decide you want to make a foot massager so that you can have the most comfortable sleep ever. But you have very particular requirements on the speeds that various parts move at, for maximum comfort. You want wheel D to move at 20rpm, E to move at 100rpm, F to move at 200rpm and G to move at 400rpm. Remember that wheel A turns at 20rpm.



But you don't have many parts, you have the following wheels on hand:



Which wheels do you put in which places to get the required speeds? (Put the wheel number in the corresponding box. The first has been done for you).

# 3 A	E
B	F
C	G
D	