Problem 1: Mental Math
NO CALCULATORS.

Example:

Question 1.1:

Question 1.2:

Question 1.3:

Question 1.4:

Question 1.5:

Question 1.6:

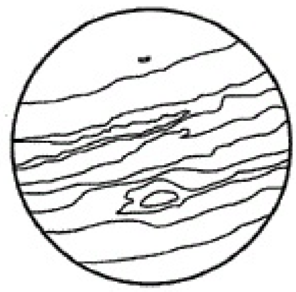
Question 1.7:

Question 1.8:

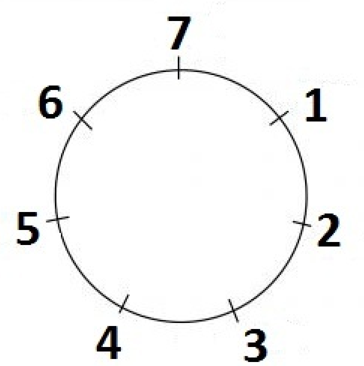
Question 1.9:

Question 1.10:

Problem 2: Time on Planet Epsilon
NO CALCULATORS. 10 MINUTES.



The inhabitants of planet Epsilon keep time a bit differently than us Earthlings. Because of the distance of planet Epsilon to its sun, one day is only 21 hours long. Also, since the Epsilonians don't like big numbers, their clocks only go up to 7, like this:



Then it is 1 o'clock *three* times a day on planet Epsilon. They distinguish the different 1 o'clocks by calling them MT, AT, and NT (for morning time, afternoon time, and night time).

For example, if it is 5 MT on planet Epsilon, after 8 hours pass it will be... 6 AT.

Part 1:

Our alien friend Omicron lives on planet Epsilon.

Question 2.1: (1 points) Omicron wakes up at 6 MT. He has to be at work at 1 AT. How much time does he have before work?

hours

Question 2.2: (1 points) Omicron gets to work on time, at 1 AT. He works for 8 hours. What time is his workday over? Write your answer using Epsilon time by giving a number and either MT, AT, or NT.

Question 2.3: (1 points) Omicron gets home and eats his dinner, and goes to bed at 5 NT. If he wakes up at 6 MT again, how many hours of sleep does Omicron get?

hours

Part 2:

Iota also lives on planet Epsilon. As it turns out, Iota is about to leave for a big vacation! For each of the following, give your answer in Epsilon time by using a number and either MT, AT, or NT.

Question 2.4: (1 points) As soon as she wakes up at 6 MT, Iota hops in her hovership and heads for the beach at the Delta Sea. It's a long trip that will take her 10 hours. What time does she arrive?

Question 2.5: (1 points) After arriving, Iota stays on her vacation for exactly 65 hours before getting back in her hovership to leave. What time does Iota leave?

Question 2.6: (1 points) What time does she get back home, given that she hits traffic on the hyperway and her trip home takes 11 hours?

Part 3:

Suppose that it is midnight (or 12 AM) on Earth, and that it also happens to be midnight (or 7 MT) on planet Epsilon. Note that on Earth, 1 AM is the hour after 12 AM. On Epsilon, similarly, 1 MT is the hour after 7 MT.

Question 2.7: (1 points) If twelve hours pass, it will be noon on Earth. What time would it be on planet Epsilon? Give your answer in Epsilon time by using a number and either MT, AT, or NT.

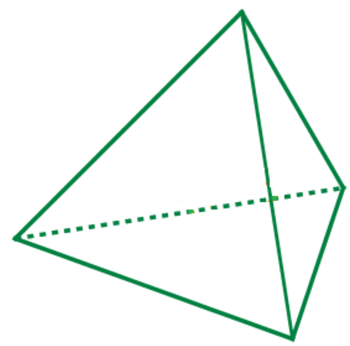
Question 2.8: (1 points) If twelve *more* hours pass, it would be midnight again on Earth. What time would it be on planet Epsilon? Give your answer in Epsilon time by using a number and either MT, AT, or NT.

Question 2.9: (2 points) Starting from when it is midnight on both Earth and Epsilon, how many hours in total must pass before the next time that midnight again occurs at the same time on both planets?

hours

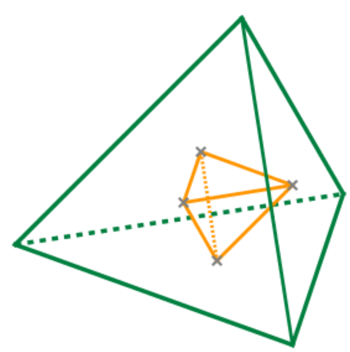
Problem 3: Dual Polyhedra
NO CALCULATORS. 10 MINUTES.

A **polyhedron** is a 3-dimensional shape with flat **faces**, straight **edges**, and sharp corners called **vertices**. The tetrahedron is an example of a polyhedron.



Every polyhedron has another polyhedron naturally related to it, which we call its **dual**. We can construct the dual as follows. Draw a vertex in the middle of each face of the original polyhedron. If two faces share an edge, draw an edge between the vertices in the middle of those faces. These vertices and edges, and the faces between them, describe the dual.

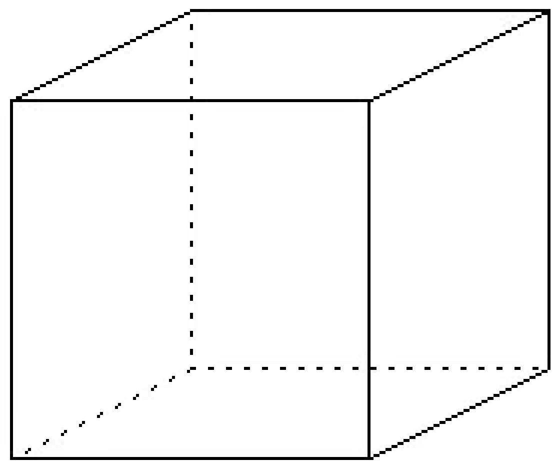
Let's do this with the tetrahedron.



The dual is the smaller shape inside the original tetrahedron — we see that the dual of the tetrahedron is another tetrahedron. Be careful! This is a special case. In general, the dual might be different from the original polyhedron.

Part 1:

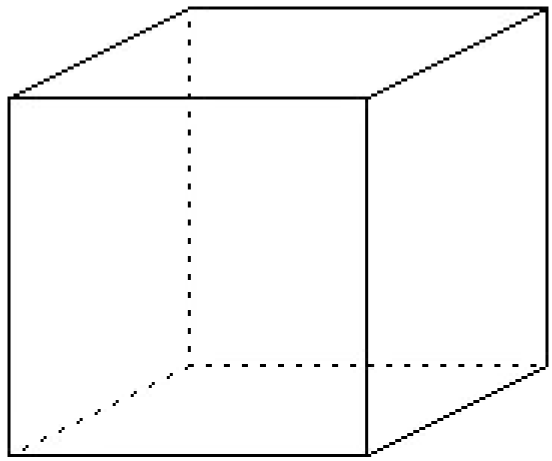
Consider the cube.



Question 3.1: (1 points) How many vertices does the cube have? How many faces? How many edges?

vertices
faces
edges

In the cube below, draw a picture of the dual of the cube. This picture will help you answer the next few questions, but itself isn't worth any points. (So don't worry *too* much about your drawing!)



Question 3.2: (1 points) What shape are the faces of the dual of the cube?

Question 3.3: (1 points) How many vertices does the dual of the cube have? How many faces? How many edges?

vertices
faces
edges

Part 2:

The **Euler characteristic** of a polyhedron is the number given by the formula

$$\big(\text{number of vertices}\big) - \big(\text{number of edges}\big) + \big(\text{number of faces}\big).$$

The tetrahedron, as on the first page, has 4 vertices, 6 edges, and 4 faces, so the Euler characteristic of the tetrahedron is

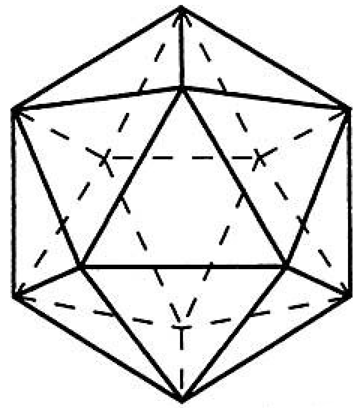
$$4 - 6 + 4 = 2.$$

Question 3.4: (1 points) What's the Euler characteristic of the cube?

Question 3.5: (1 points) What's the Euler characteristic of the *dual* of the cube?

Part 3:

The following poyhedron is called an icosahedron. It has 20 triangular faces. At each of the 12 vertices, 5 of the faces meet. The Euler characteristic of the icosahedron is 2.



Question 3.6: (1 points) How many vertices does the *dual* of the icosahedron have?

Question 3.7: (2 points) What shape are the faces of the *dual* of the icosahedron?

Question 3.8: (2 points) The icosahedron has the same Euler characteristic as its dual. The icosahedron also has the same number of edges as its dual. How many faces does the *dual* of the icosahedron have?

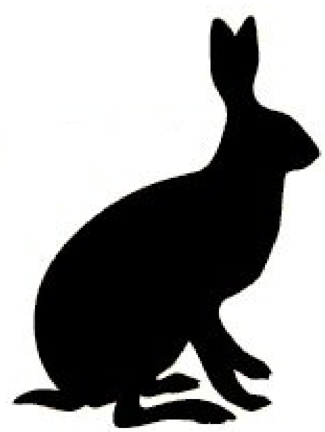
Problem 4: Chickens and Rabbits

NO CALCULATORS. 15 MINUTES.

Michael is helping out at his grandfather's animal farm. Michael's grandfather has chickens and rabbits, which are kept in pens. Today, Michael finds only one pen available, so he has to put chickens and rabbits in the same pen.

Grandpa asks: "Michael, can you tell me how many animal feet are there in the pen, by adding up the chickens' feet and rabbits' feet?"
Michael counts and answers: "20!"
Grandpa: "How many animal heads do we have?"
Michael: "7!"
Grandpa: "Good job kid. So we have 4 chickens and 3 rabbits!"

Michael was so surprised, as he couldn't believe Grandpa could know the number of chickens and rabbits without counting their bodies. Do you know how Grandpa did it?



Part 1:

Question 4.1: (2 points) If there are 12 chickens and 23 rabbits in a pen, how many heads and how many feet are in the pen?

heads
feet

Question 4.2: (2 points) Michael moves on to a different pen, which also contains chickens and rabbits. Which of the following numbers could not be the number of the feet in this pen? In your answer, give *all* the numbers which cannot be the number of feet in the pen.
A. 45
B. 44
C. 43
D. 2014

Part 2:

But how did Grandpa solve for the numbers of chickens and rabbits from just the numbers of heads and feet? Let's see one more example.

Q: There are 2 heads and 6 feet in the pen. How many chickens and rabbits are there in the pen?
A: Let's think about it this way: if the 2 heads are both chickens' heads, then 2 chickens will have $2 \times 2 = 4$ feet, which is less than 6.

Now, one rabbit has two more feet than one chicken, so let's make one of the chickens a rabbit instead. Then we still have the same number of heads, but two more feet: we have 2 heads and 6 feet in total, like we wanted. So there must be one chicken and one rabbit in the pen.

Question 4.3: (2 points) If the total number of feet is 44 and the total number of heads is 15, how many chickens and how many rabbits are there in the pen?

chickens
rabbits

Part 3:

Suppose we are on another planet again! This planet is called Pandora. You are given a similar task: using only the total number of heads and feet, calculate the number of Pandora chickens (called Pickens) and Pandora rabbits (called Pabbits) in one pen. A Picken has 1 head and 3 feet; a Pabbit has 1 head and 7 feet.

Question 4.4: (2 points) Suppose we know the total number of heads is 18 and the total number of feet is 82. How many Pickens and how many Pabbits are there in the pen?

Pickens
Pabbits

Question 4.5: (2 points) Instead, suppose we know the total number of heads is 180 and the total number of feet is 832. How many Pickens and how many Pabbits are there in the pen? *Hint: your previous answer may help get you started.*

Pickens
Pabbits

Problem 5: Double Secret Message
NO CALCULATORS. TEAM PROBLEM. 30 MINUTES.

You have found a secret message that seems to come in two parts... can you decode the double secret message?

Part 1: (20 points)

The first part of the message is encoded using a **Caesar shift**. In a Caesar shift, each letter has been replaced with a different letter by shifting the alphabet to the right. For example, in a Caesar shift of 3, every A in the message is replaced with D, every B with E, and so on until every Y is replaced with B and every Z is replaced with C.

The first message is:

> cqn mxdkun bnlanc vnbbjpn fjb nwlxmnm rw cqn oxuuxfrwp fjh: orabc njlq unccna fjb cdawnm rwcx j wdvkna (j = 0,k = 1, ..., i=25). wngc, njlq wdvkna fjb vducryurnm kh 3 jwm 10 fjb jmmnm. cqnw, njlq wdvkna fjb lxwenacnm cx cqn bjvn wdvkna vxmdux 26. orwjuuh, cqn wnf wdvknab fnan lxwenacnm kjlt rwcx unccnab.

Hint: the most commonly used letter in English is "e".

Part 2: (30 points)

It appears that the first part of the message contains the instructions needed to *encode* the second part! To solve this part of the problem, you will need to *reverse the process* used to encode the second message in order to decode it.

You will also need to know that

$$a \quad modulo \quad b$$

is the remainder left when a is divided by b. For example,

$$14 \text{ modulo } 4 = 2$$

because

$$14 \div 4 = 3 \text{ remainder } 2.$$

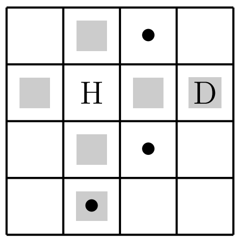
The second message is:

ax yimqaxmix

Problem 6: Heron, Dingo, Badger

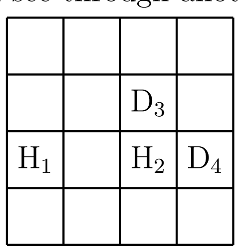
NO CALCULATORS. TEAM PROBLEM. 30 MINUTES.

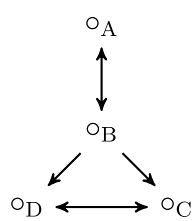
On Planet Flagellan there is a large meadow where Badgers and Dingoes and Herons all live together. These animals hardly ever move, and some Flagellans even make maps showing the positions of the animals. The animals also have *directional vision.* For example, Herons can only see along straight lines in the horizontal and vertical directions, while Dingoes can see only along diagonals. Let's look at some maps now. The first map shows a Heron and a Dingo.



In the map, the spaces the Heron can see are shaded, while the spaces the Dingo can see have dots. To the right of the map is something called a *Sight-Graph* containing two dots and an arrow. The two dots represent the two animals in the map, and the arrow points from one animal to another that it can see. By examining the map we notice that the Heron can see the Dingo but not vice versa. This is why the arrow only goes in one direction. The Heron must correspond to the upper dot in the Sight-Graph. The locations of the dots do not matter (an "upper" dot can represent an animal on the bottom part of the map).

Next, here are two Herons and two Dingoes, labeled with numbers. No animal can see through another animal, so H_1 can't see D_4.





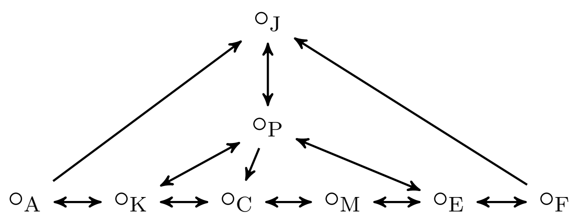
This Sight-Graph has four dots representing the four animals. Let's figure out how it relates to the map. For example, H_2 can see all of the other animals on the map, so it must go with the middle dot, B. Then since H_1 can see H_2, we know that H_1 goes with dot A. Finally, C and D must be the two Dingos, although we can't tell which is which.

Badgers have the strongest vision: they can see vertically and horizontally like Herons, and also diagonally like Dingoes.

Part 1: Warmup: Maps and Graphs

Question 6.1: (10 points) Here is a map with a corresponding Sight-Graph. The animals are labeled with numbers and the dots are labeled with letters.

D_8			H_6
	B_9		H_5
H_1	H_2	H_3	H_4



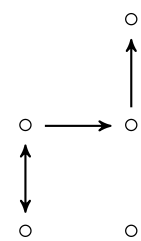
Which animal goes with dot J?

Which dot goes with the Heron H_4?

Part 2: Completions

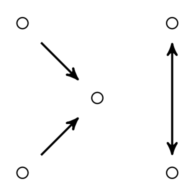
Question 6.2: (8 points) Add some arrows to complete the Sight-Graph. *Hint: On the map, only one pair of animals can see each other.*

H		H	
	D	D	
			H



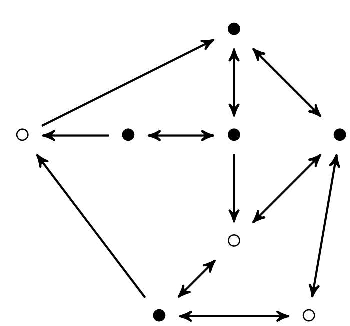
Question 6.3: (8 points) Add two Herons to the map so that the Sight-Graph is correct. *Hint: In the sight graph there is a two-way arrow. What part of the map corresponds to that?*

		D	
		D	
			D



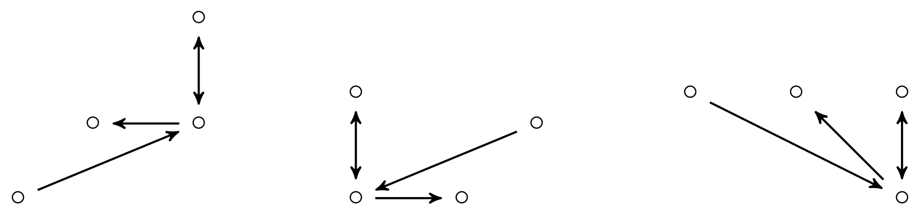
Question 6.4: (9 points) Add three animals to the map so that the Sight-Graph is correct. The five solid dots in the Sight-Graph represent the animals already on the map. *Hint: one of the five solid dots has no arrows to or from the other four solid dots.*

H			B
H			
H			
B			



Part 3: Which are the same?

Remember that the locations of dots in a Sight-Graph do not matter. This means that these three Sight-Graphs are all the same, even if they look a little different:



Another way to say this is that *two Sight-Graphs are the same if you can move the dots in one to make it look like the other one.* The arrows stay attached to the dots while they move.

Question 6.5: (7 points) Here are two maps. Are their Sight-Graphs the same? (Yes or No)

H		H	
H		B	
		D	

	D		
D	H		D
	D		

Question 6.6: (8 points) Here are two maps. Are their Sight-Graphs the same? (Yes or No)

	H			
	B			
H	H			
	H			D

	D	D	H
H	H		B